# A More General Maximal Bernstein-type Inequality

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May 24, 2012

**Abstract**

We extend a general Bernstein-type maximal inequality of Kevei and Mason (2011) for sums of random variables.

*Keywords:* Bernstein inequality, dependent sums, maximal inequality, mixing, partial sums.

*AMS Subject Classificiation:* MSC 60E15; MSC 60F05; MSC 60G10.

## 1 Introduction

Let $X_1, X_2, \ldots$ be a sequence of random variables, and for any choice of $1 \le k \le l < \infty$ we denote the partial sum $S(k,l) = \sum_{i=k}^{l} X_i$, and define $M(k,l) = \max\{|S(k,k)|, \ldots, |S(k,l)|\}$. It turns out that under a variety of assumptions the partial sums $S(k,l)$ will satisfy a generalized Bernstein-type inequality of the following form: for suitable constants $A > 0$, $a > 0$, $b \ge 0$ and $0 < \gamma < 2$ for all $m \ge 0$, $n \ge 1$ and $t \ge 0$,

$$\mathbf{P}\{|S(m+1, m+n)| > t\} \le A \exp\left\{ -\frac{at^2}{n + bt^\gamma} \right\}. \tag{1.1}$$

Kevei and Mason [2] provide numerous examples of sequences of random variables $X_1, X_2, \ldots,$ that satisfy a Bernstein-type inequality of the form (1.1). They show, somewhat unexpectedly, without any additional assumptions, a modified version of it also holds for $M(1+m, n+m)$ for all $m \ge 0$ and $n \ge 1$. Here is their main result.

**Theorem 1.1.** *Assume that for constants $A > 0$, $a > 0$, $b \ge 0$ and $\gamma \in (0,2)$, inequality (1.1) holds for all $m \ge 0, n \ge 1$ and $t \ge 0$. Then for every $0 < c < a$ there exists a $C > 0$ depending only on $A, a$, $b$ and $\gamma$ such that for all $n \ge 1$, $m \ge 0$ and $t \ge 0$,*

$$\mathbf{P}\{M(m+1, m+n) > t\} \le C \exp\left\{ -\frac{ct^2}{n + bt^\gamma} \right\}. \tag{1.2}$$

*Supported by the TAMOP–4.2.1/B–09/1/KONV–2010–0005 project.
†Research partially supported by NSF Grant DMS–0503908.

There exists an interesting class of Bernstein-type inequalities that are not of the form (1.1). Here are two motivating examples.

**Example 1.** Assume that $X_1, X_2, \ldots,$ is a stationary Markov chain satisfying the conditions of Theorem 6 of Adamczak [1] and let $f$ be any bounded measurable function such that $Ef(X_1) = 0$. His theorem implies that for some constants $D > 0$, $d_1 > 0$ and $d_2 > 0$ for all $t \geq 0$ and $n \geq 1$,

$$P\{|S_n(f)| \geq t\} \leq D^{-1} \exp\left(-\frac{Dt^2}{nd_1 + td_2 \log n}\right), \tag{1.3}$$

where $S_n(f) = \sum_{i=1}^n f(X_i)$, and $D/d_1$ is related to the limiting variance in the central limit theorem.

**Example 2.** Assume that $X_1, X_2, \ldots,$ is a strong mixing sequence with mixing coefficients $\alpha(n)$, $n \geq 1$, satisfying for some $d > 0$, $\alpha(n) \leq \exp(-2dn)$. Also assume that $EX_i = 0$ and for some $M > 0$, $|X_i| \leq M$, for all $i \geq 1$. Theorem 2 of Merlevède, Peligrad and Rio [4] implies that for some constant $D > 0$ for all $t \geq 0$ and $n \geq 1$,

$$P\{|S_n| \geq t\} \leq D \exp\left(-\frac{Dt^2}{nv^2 + M^2 + tM(\log n)^2}\right), \tag{1.4}$$

where $S_n = \sum_{i=1}^n X_i$ and $v^2 = \sup_{i>0}\left(Var(X_i) + 2\sum_{j>i}|cov(X_i, X_j)|\right)$.

The purpose of this note to establish the following extended version of Theorem 1.1 that will show that a maximal version of inequalities (1.3) and (1.4) also holds.

**Theorem 1.2.** *Assume that there exist constants $A > 0$ and $a > 0$ and a sequence of non-decreasing non-negative functions $\{g_n\}_{n\geq 1}$ on $(0, \infty)$, such that for all $t > 0$ and $n \geq 1$, $g_n(t) \leq g_{n+1}(t)$ and for all $0 < \rho < 1$*

$$\lim_{n\to\infty} \inf\left\{\frac{t^2}{g_n(t)\log t} : g_n(t) > \rho n\right\} = \infty, \tag{1.5}$$

*where the infimum of the empty set is defined to be infinity, such that for all $m \geq 0$, $n \geq 1$ and $t \geq 0$,*

$$P\{|S(m+1, m+n)| > t\} \leq A \exp\left\{-\frac{at^2}{n + g_n(t)}\right\}. \tag{1.6}$$

*Then for every $0 < c < a$ there exists a $C > 0$ depending only on $A, a$ and $\{g_n\}_{n\geq 1}$ such that for all $n \geq 1$, $m \geq 0$ and $t \geq 0$,*

$$P\{M(m+1, m+n) > t\} \leq C \exp\left\{-\frac{ct^2}{n + g_n(t)}\right\}. \tag{1.7}$$

Note that condition (1.5) trivially holds when the functions $g_n$ are bounded, since the corresponding sets are empty sets. However, in the interesting cases $g_n$'s are not bounded, and in this case the condition basically says that $g_n(t)$ increases slower than $t^2$.

Essentially the same proof shows that the statement of Theorem 1.2 remains true if in the numerator of (1.6) and (1.7) the function $t^2$ is replaced by a regularly varying function at infinity $f(t)$ with a positive index. In this case the $t^2$ in condition (1.5) must be replaced by $f(t)$. Since we do not know any application of a result of this type, we only mention this generalization.

*Proof.* Choose any $0 < c < a$. We prove our theorem by induction on $n$. Notice that by the assumption, for any integer $n_0 \geq 1$ we may choose $C > An_0$ to make the statement true for all $1 \leq n \leq n_0$. This remark will be important, because at some steps of the proof we assume that $n$ is large enough. Also since the constants $A$ and $a$ in (1.6) are independent of $m$, we can without loss of generality assume $m = 0$.
Assume the statement holds up to some $n \geq 2$. (The constant $C$ will be determined in the course of the proof.)
**Case 1.** Fix a $t > 0$ and assume that

$$g_{n+1}(t) \leq \alpha\, n, \tag{1.8}$$

for some $0 < \alpha < 1$ be specified later. (In any case, we assume that $\alpha n \geq 1$.) Using an idea of [5], we may write for arbitrary $1 \leq k < n$, $0 < q < 1$ and $p + q = 1$ the inequality

$$\begin{aligned} P\{M(1, n+1) > t\} \leq & P\{M(1, k) > t\} + P\{|S(1, k+1)| > pt\} \\ & + P\{M(k+2, n+1) > qt\}. \end{aligned}$$

Let

$$u = \frac{n + g_{n+1}(qt) - q^2 g_{n+1}(t)}{1 + q^2}.$$

Note that $u \leq n - 1$ if $0 < \alpha < 1$ is chosen small enough depending on $q$, for $n$ large enough. Notice that

$$\frac{t^2}{u + g_{n+1}(t)} = \frac{q^2 t^2}{n - u + g_{n+1}(qt)}. \tag{1.9}$$

Set

$$k = \lceil u \rceil\,. \tag{1.10}$$

Using the induction hypothesis and (1.6), keeping in mind that $1 \leq k \leq n - 1$, we obtain

$$\begin{aligned} P\{M(1, n+1) > t\} \leq & C \exp\left\{-\frac{ct^2}{k + g_k(t)}\right\} + A \exp\left\{-\frac{ap^2t^2}{k + 1 + g_{k+1}(pt)}\right\} \\ & + C \exp\left\{-\frac{cq^2t^2}{n - k + g_{n-k}(qt)}\right\} \\ \leq & C \exp\left\{-\frac{ct^2}{k + g_{n+1}(t)}\right\} + A \exp\left\{-\frac{ap^2t^2}{k + 1 + g_{n+1}(pt)}\right\} \\ & + C \exp\left\{-\frac{cq^2t^2}{n - k + g_{n+1}(qt)}\right\}. \end{aligned} \tag{1.11}$$

Notice that we chose $k$ to make the first and third terms in (1.11) almost equal, and since by (1.10)

$$\frac{t^2}{k + g_{n+1}(t)} \leq \frac{q^2 t^2}{n - k + g_{n+1}(qt)}$$

the first term is greater than or equal to the third.
First we handle the second term in formula (1.11), showing that whenever $g_{n+1}(t) \leq \alpha n$,

$$\exp\left\{-\frac{ap^2t^2}{k + 1 + g_{n+1}(pt)}\right\} \leq \exp\left\{-\frac{ct^2}{n + 1 + g_{n+1}(t)}\right\}.$$

For this we need to verify that for $g_{n+1}(t) \le \alpha n$,

$$\frac{ap^2}{k+1+g_{n+1}(pt)} > \frac{c}{n+1+g_{n+1}(t)}, \tag{1.12}$$

which is equivalent to

$$ap^2(n+1+g_{n+1}(t)) > c(k+1+g_{n+1}(pt)).$$

Using that

$$k = \lceil u \rceil \le u+1 = 1 + \frac{1}{1+q^2}\left[n + g_{n+1}(qt) - q^2 g_{n+1}(t)\right],$$

it is enough to show

$$\begin{aligned} & n\left(ap^2 - \frac{c}{1+q^2}\right) + ap^2 - 2c \\ + & \left[g_{n+1}(t)ap^2 - g_{n+1}(pt)c - \frac{c}{1+q^2}\left(g_{n+1}(qt) - q^2 g_{n+1}(t)\right)\right] > 0. \end{aligned}$$

Note that if the coefficient of $n$ is positive, then we can choose $\alpha$ in (1.8) small enough to make the above inequality hold. So in order to guarantee (1.12) (at least for large $n$) we only have to choose the parameter $p$ so that $ap^2 - c > 0$, which implies that

$$ap^2 - \frac{c}{1+q^2} > 0 \tag{1.13}$$

holds, and then select $\alpha$ small enough, keeping mind that we assume $\alpha n \ge 1$ and $k \le n-1$.
Next we treat the first and third terms in (1.11). Because of the remark above, it is enough to handle the first term. Let us examine the ratio of $C\exp\{-ct^2/(k+g_{n+1}(t))\}$ and $C\exp\{-ct^2/(n+1+g_{n+1}(t))\}$. Notice again that since $u+1 \ge k$, the monotonicity of $g_{n+1}(t)$ and $g_{n+1}(t) \le \alpha n$ implies

$$\begin{aligned} n+1-k \ge n-u &= n - \frac{n + g_{n+1}(qt) - q^2 g_{n+1}(t)}{1+q^2} \\ &\ge \frac{q^2 n - (1-q^2)g_{n+1}(t)}{1+q^2} \\ &\ge n\frac{q^2 - \alpha(1-q^2)}{1+q^2} \\ &=: c_1 n. \end{aligned}$$

At this point we need that $0 < c_1 < 1$. Thus we choose $\alpha$ small enough so that

$$q^2 - \alpha(1-q^2) > 0. \tag{1.14}$$

Also we get using $g_{n+1}(t) \le \alpha n$ the bound

$$(n+1+g_{n+1}(t))(k+g_{n+1}(t)) \le 2n^2(1+\alpha)^2 =: c_2 n^2,$$

which holds if $n$ large enough. Therefore, we obtain for the ratio

$$\exp\left\{-ct^2\left(\frac{1}{k+g_{n+1}(t)} - \frac{1}{n+1+g_{n+1}(t)}\right)\right\} \le \exp\left\{-\frac{cc_1 t^2}{c_2 n}\right\} \le \mathrm{e}^{-1},$$

whenever $cc_1 t^2/(c_2 n) \geq 1$, that is $t \geq \sqrt{c_2 n/(cc_1)}$. Substituting back into (1.11), for $t \geq \sqrt{c_2 n/(cc_1)}$ and $g_{n+1}(t) \leq \alpha n$ we obtain

$$
\begin{aligned}
& P\{M(1, n+1) > t\} \\
& \leq \left(\frac{2}{\mathrm{e}} C + A\right) \exp\{-ct^2/(n+1+g_{n+1}(t))\} \leq C \exp\{-ct^2/(n+1+g_{n+1}(t))\},
\end{aligned}
$$

where the last inequality holds for $C > A\mathrm{e}/(\mathrm{e}-2)$.
Next assume that $t < \sqrt{c_2 n/(cc_1)}$. In this case choosing $C$ large enough we can make the bound $> 1$, namely

$$
C \exp\left\{-\frac{ct^2}{n+1+g_{n+1}(t)}\right\} \geq C \exp\left\{-\frac{cc_2 n}{cc_1 n}\right\} = C\mathrm{e}^{-c_2/c_1} \geq 1,
$$

if $C > \mathrm{e}^{c_2/c_1}$.

**Case 2.** Now we must handle the case $g_{n+1}(t) > \alpha n$. Here we apply the inequality

$$
P\{M(1, n+1) > t\} \leq P\{M(1, n) > t\} + P\{|S(1, n+1)| > t\}.
$$

Using assumption (1.6) and the induction hypothesis, we have

$$
\begin{aligned}
P\{M(1, n+1) > t\} &\leq C \exp\left\{-\frac{ct^2}{n + g_n(t)}\right\} + A \exp\left\{-\frac{at^2}{n+1+g_{n+1}(t)}\right\} \\
&\leq C \exp\left\{-\frac{ct^2}{n + g_{n+1}(t)}\right\} + A \exp\left\{-\frac{at^2}{n+1+g_{n+1}(t)}\right\}.
\end{aligned}
$$

We will show that the right side $\leq C \exp\{-ct^2/(n+1+g_{n+1}(t))\}$. For this it is enough to prove

$$
\begin{aligned}
& \exp\left\{-ct^2 \left(\frac{1}{n+g_{n+1}(t)} - \frac{1}{n+1+g_{n+1}(t)}\right)\right\} \\
& + \frac{A}{C} \exp\left\{-\frac{t^2(a-c)}{n+1+g_{n+1}(t)}\right\} \leq 1.
\end{aligned} \tag{1.15}
$$

Using the bound following from $g_{n+1}(t) > \alpha n$ and recalling that $\alpha n \geq 1$ and $0 < \alpha < 1$, we get

$$
\frac{t^2}{(n+g_{n+1}(t))(n+1+g_{n+1}(t))} \geq \frac{\alpha^2 t^2}{(1+\alpha)(1+2\alpha) g_{n+1}(t)^2} =: c_3 \frac{t^2}{g_{n+1}(t)^2},
$$

and

$$
\frac{t^2(a-c)}{n+1+g_{n+1}(t)} \geq \frac{t^2}{g_{n+1}(t)} \frac{\alpha(a-c)}{1+2\alpha} =: \frac{t^2}{g_{n+1}(t)} c_4.
$$

Choose $\delta > 0$ so small such that $0 < x \leq \delta$ implies $\mathrm{e}^{-cc_3 x^2} \leq 1 - \frac{cc_3}{2} x^2$.
For $t/g_{n+1}(t) \geq \delta$ the left-hand side of (1.15) is less then

$$
\mathrm{e}^{-cc_3 \delta^2} + \frac{A}{C},
$$

which is less than 1, for $C$ large enough.

For $t/g_{n+1}(t) \le \delta$ by the choice of $\delta$ the left-hand side of (1.15) is less then

$$1 - \frac{cc_3}{2}\frac{t^2}{g_{n+1}(t)^2} + \frac{A}{C}\exp\left\{-\frac{t^2}{g_{n+1}(t)}c_4\right\},$$

which is less than 1 if

$$\frac{cc_3}{2}\frac{t^2}{g_{n+1}(t)^2} > \frac{A}{C}\exp\left\{-\frac{t^2}{g_{n+1}(t)}c_4\right\}.$$

By (1.5), for any $0 < \eta < 1$ and all large enough $n$, $g_{n+1}(t)1\left\{g_{n+1}(t) > \alpha n\right\} \le \eta t^2$, so that for all large $n$, whenever $g_{n+1}(t) > \alpha n$, we have

$$\frac{t^2}{g_{n+1}(t)^2} \ge t^{-2},$$

and again by (1.5) for all large $n$, whenever $g_{n+1}(t) > \alpha n$, $t^2/g_{n+1}(t) \ge (3/c_4)\log t$. Therefore for all large $n$, whenever $g_{n+1}(t)\, \alpha n$,

$$\exp\left\{-\frac{t^2}{g_{n+1}(t)}c_4\right\} \le t^{-3},$$

which is smaller than $t^{-2}\frac{Ccc_3}{2A}$, for $t$ large enough, i.e. for $n$ large enough. The proof is complete. $\square$

By choosing $g_n(t) = bt^\gamma$ for all $n \ge 1$ we see that Theorem 1.2 gives Theorem 1.1 as a special case. Also note that Theorem 1.2 remains valid for sums of Banach space valued random variables with absolute value $|\cdot|$ replaced by norm $||\cdot||$. Theorem 1.2 permits us to derive the following maximal versions of inequalities (1.3) and (1.4).

**Application 1**. In Example 1 one readily checks that the assumptions of Theorem 1.2 are satisfied with $A = D^{-1}$ and $a = D/d_1$

$$g_n(t) = \left(\frac{td_2}{d_1}\right)\log n.$$

We get the maximal version of inequality (1.3) holding for any $0 < c < 1$ and all $n \ge 1$ and $t > 0$

$$P\left\{\left|\max_{1\le m\le n} S_n(f)\right| \ge t\right\} \le C\exp\left(-\frac{cDt^2}{nd_1 + td_2\log n}\right), \tag{1.16}$$

for some constant $C \ge D^{-1}$ depending on $c$, $D^{-1}$, $D/d_1$ and $\{g_n\}_{n\ge1}$.

**Application 2.** In Example 2 one can verify that the assumptions of the Theorem 1.2 hold with $A = D$ and $a = D/v^2$ and

$$g_n(t) = \frac{M^2}{v^2} + \left(\frac{tM}{v^2}\right)(\log n)^2,$$

which leads to the maximal version of inequality (1.4) valid for any $0 < c < 1$ and all $n \ge 1$ and $t > 0$

$$P\left\{\max_{1\le m\le n}|S_m| \ge t\right\} \le C\exp\left(-\frac{cDt^2}{nv^2 + M^2 + tM(\log n)^2}\right) \tag{1.17}$$

for some constant $C \geq D$ depending on $c$, $D/v^2$ and $\{g_n\}_{n\geq 1}$. See Corollary 24 of Merlevède and Peligrad [3] for a closely related inequality that holds for all $n \geq 2$ and $t > K \log n$ for some $K > 0$.

**Remark** There is a small oversight in the published version of the Kevei and Mason paper. Here are the corrections that fix it.

1. Page 1057, line -9: Replace "$1 \leq k \leq n$" by "$1 \leq k < n$".
2. Page 1057, line -7: Replace this line with
$\leq \mathbf{P}\{M(1,k) > t\} + \mathbf{P}\{S(1,k+1) > pt\} + \mathbf{P}\{M(k+2,n+1) > qt\}$.
3. Page 1058: Replace "$k + bp^\gamma t^\gamma$" by "$k + 1 + bp^\gamma t^\gamma$" in equations (2.4) and (2.5), as well as in line -13.
4. Page 1058: Replace "$ap^2 - c$" by "$ap^2 - 2c$" in line -9.

### Acknowledgment

We thank a referee for a careful reading of the manuscript and a number of useful comments.

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